AE2403 Vibrations And Elements of Aeroelasticity

UNIT – I

BASIC NOTIONS

Vibration is a repetitive, periodic, or oscillatory response of a mechanical system. The rate of the vibration cycles is termed "frequency." Repetitive motions that are somewhat clean and regular, and that occur at relatively low frequencies, are commonly called oscillations, while any repetitive motion, even at high frequencies, with low amplitudes, and having irregular and random behaviour falls into the general class of vibration. Vibrations can naturally occur in an engineering system and may be representative of its free and natural dynamic behaviour. Also, vibrations may be forced onto a system through some form of excitation. The excitation forces may be either generated internally within the dynamic system, or transmitted to the system through an external source. When the frequency of the forcing excitation coincides with that of the natural motion, the system will respond more vigorously with increased amplitude. This condition is known as resonance, and the associated frequency is called the resonant frequency. There are "good vibrations," which serve a useful purpose. Also, there are "bad vibrations," which can be unpleasant or harmful. For many engineering systems, operation at resonance would be undesirable and could be destructive. Suppression or elimination of bad vibrations and generation of desired forms and levels of good vibration are general goals of vibration engineering.

Introduction:

Vibration is defined as a motion which repeats after equal interval of time and is also a periodic motion. The swinging of a pendulum is a simple example of vibration. Vibration occurs in all bodies which are having mass and elasticity. They are caused due to several reasons such as presence of unbalanced force in rotating machines, elastic nature of the system, external application of force or wind load sand earthquakes. Vibrations are undesirable as they induce high stresses in system components leading to noise and failure, in such cases they are to be minimized if not totally eliminated.. The desirable effects are seen in musical instruments and cement compactors used in construction work. From subject point of view the following notations and definitions are very important:

Periodic Motion:

It is a motion which repeats itself after equal intervals of time, e.g., the oscillations of simple pendulum

Time Period (T) :

It is the time required for one complete cycle or to and fro motion.The unit is seconds.

Frequency (f or ω) :

It is the number of cycles per unit time. The unit are radians/sec. or Hz.

Amplitude (X or A) :

It is the displacement of a vibrating body from its equilibrium position. It has units of length in general.

Natural Frequency (fn):

It is the frequency with which a body vibrates when subjected to an initial external disturbance and allowed to vibrate without external force being applied subsequently.

Fundamental Mode of Vibration:

A vibrating body may have more than one natural frequency and when it vibrates with the lowest natural frequency ,it is the Fundamental mode of vibration.

Degrees of Freedom:

It is the minimum number of coordinates required to describe the motion of system. Typically in our discussions 1DOF system will have one mass, e.g., a spring attached with one mass, 2 DOF systems will have two masses and likewise we have 3DOFsystem. A continuous system like a beam or plate consisting of infinite number of particles with mass, are systems with infinite number of DOF.

Simple Harmonic Motion (SHM):

It is a periodic motion with acceleration always directed towards the equilibrium position. It can also be defined as projection of motion of a particle along a circle with uniform angular velocity on the diameter of circle.

Damping:

It is the resistance offered to the motion of a vibrating body by absorbing the energy of vibrations. Such vibrations are termed as damped vibrations.

Forced Vibrations:

It is the vibration of a body when subjected to an external force which is periodic in nature and vibrations occur as long as external force is present.

Resonance:

It is said to occur in the system when the amplitude of vibrations are excessive leading to failure. This occurs in forced vibrations when the frequency of externally applied force is same as that of natural frequency of the body.

Linear and Non Linear Vibrations:

When the vibrations are represented by linear differential equations and laws of superposition are applicable for the system, we have linear systems. Non linear Vibrations are experienced when large amplitudes are encountered and laws of superposition are not applicable.

Longitudinal, Transverse and Torsional Vibrations:

When the motion of mass of the system is parallel to the axis of the system, we have Longitudinal vibrations. When the motion of mass is perpendicular to the system axis the vibrations are Transverse vibrations and when the mass twists and untwists about the axis the vibrations are Torsional vibrations. Up and down motion of mass in a spring mass system represents Longitudinal vibrations. Vibration of a cantilever beam represents Transverse vibrations. The twisting and untwisting of a disc attached at the end of a shaft represents Torsional vibrations.

Vector representation of SHM:

Any SHM can be represented as by the equation, $x = A \sin\omega t$---(1) , where x is the displacement , A is the amplitude , ω is the circular frequency and t is the time, Differentiating eqn.1 w.r.t. t we have velocity vector and differentiating eqn 1 twice we have the acceleration vector. If x1 and x2 are two displacement vectors withsame frequencies then the phase difference between them is given by ϕ.

Principle of Superposition:

When two SHM of same frequencies are added the resulting motion is also a harmonic motion.

Consider two harmonic motions $x1 = A1\sin\omega t$ and $x2 = A2 \sin(\omega t + \phi)$. Then if x is the resultant displacement , $x = x1 + x2$. The resultant amplitude $x = A\sin(\omega t + \theta)$, where A is the resultant amplitude and is acting at an angle θ w.r.t vector x1.The above addition of SHMs can also be done graphically.

Sample Problems:

1) Add the following harmonics analytically and check the solution graphically $x1= 3\sin(\omega t + 30^\circ)$, $x2 = 4 \cos(\omega t + 10^\circ)$

Solution:

Given : $x1= 3\sin(\omega t + 30^\circ)$, $x2 = 4 \cos(\omega t + 10^\circ)$

Analytical method:

We know that, $x = x1 + x2 = A \sin(\omega t + \theta)$

Make x1 and x2 to have same Sin terms always,

i.e., $x2 = 4 \cos(\omega t + 10^\circ + 90^\circ) = 4\sin(\omega t + 100^\circ)$

Hence,

$A \sin(\omega t + \theta) = 3 \sin(\omega t + 30^\circ) + 4 \sin(\omega t + 100^\circ)$

Expanding LHS and RHS

$A\sin\omega t \cos\theta + A \cos\omega t \sin\theta = 3 \sin\omega t \cos 30^\circ + 3\cos\omega t \sin 30^\circ + 4 \sin\omega t \cos 100^\circ + 4 \sin\omega t \sin 100^\circ$

A Sin ωt Cos θ + A cosωt Sin θ = Sin ωt(1.094) + Cos ωt (5.44)

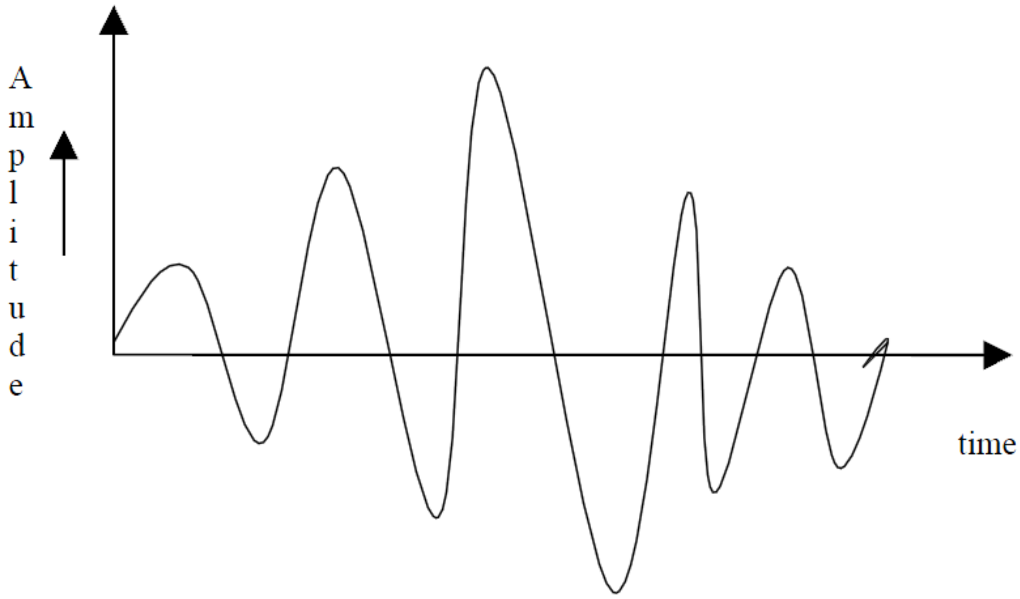
Comparing the coefficients of A Cos θ and A Sin θ in the above equation

ACos θ = 1.094, ASin θ = 5.44 , tan θ = ASin θ/ ACos θ = 5.44/1.904

Therefore, θ = 70.7 ◦ and A = 1.094/Cos 70.7 ◦ = 5.76.

Beats Phenomenon:

Consider two harmonics x1 and x2 of slightly different frequencies and the A cosϕ resulting motion will not be a SHM. Due to existence of different frequencies the phase difference of the two vectors keeps on changing and shifting w.r.t time. The two harmonics when in phase have their resultant amplitude to be sum of individual amplitudes and when they are out of phase the resultant amplitude is difference of individual amplitudes. This phenomenon of varying of resultant amplitude is called as Beats and this occurs at a frequency given by the difference of the individual frequencies of the two vectors.



Fourier Theorem:

Any periodic motion can be represented in terms of sine and cosine terms called as Fourier series. The process of obtaining the Fourier series of a periodic motion is called Harmonic analysis, i.e.,

F(t) a periodic function can be represented as

F(t) = a0 + a1cosωt + a2cos2ωt + a3cos3ωt +…….an cosn ωt + b1sinωt + b2sin2ωt + b3sin3ωt +…….. bn sin n ωt

The constants a0, a1, a2,,,, and b1, b2, b3…. etc., are obtained using the following formulae:

a0 = (ω/2π) ∫ F(t), in the limits 0 to 2π/ω
an=(ω/π) ∫ F(t)cos(nωt) dt, in the limits 0 to 2π/ω
bn= =(ω/π) ∫ F(t)sin(nωt) dt, in the limits 0 to 2π/ω

4. Represent the above periodic motion using harmonic series

Solution:

Mathematically for one complete cycle we have the eqn for AB as

x(t) = -20t +2 for 0<t<0.2

T=0.2, ω= (2π/T) = 10π

a0=(ω/2π) ∫ x(t)dt

a0= (10π/2π) ∫ (-20t +2)dt = 0

an = (ω/π) ∫ x(t)cos(nωt)dt

bn = (ω/π) ∫ x(t)sin(nωt)dt= (4/πn)

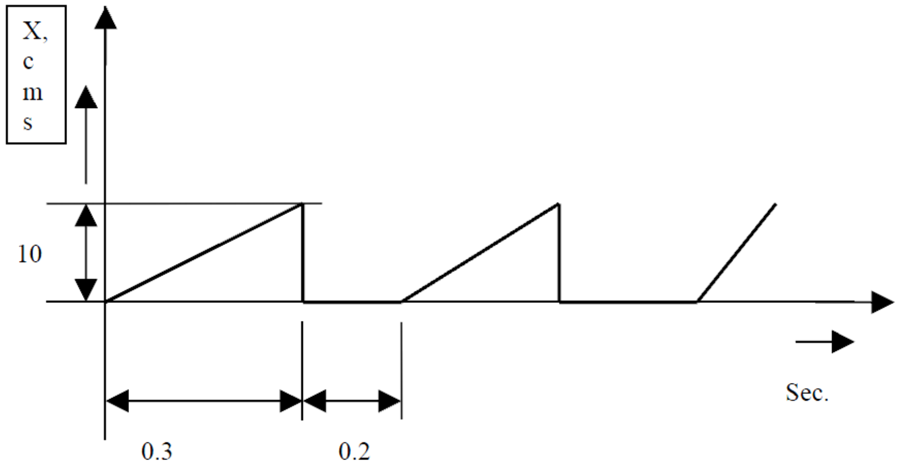
Thus, the harmonic series is,

x(t) = 4/πΣ (1/n) sin10πnt , for n =1,2,…5)

A periodic motion is represented by a saw tooth wave form, the amplitude is

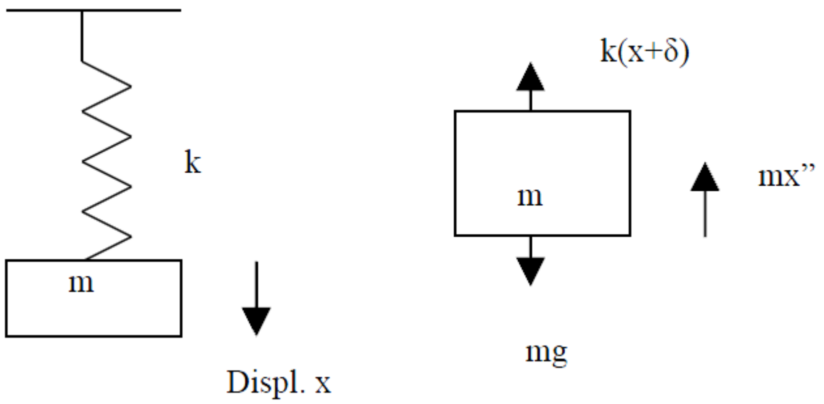
0 at t=0and rises to 10 cm, at t=0.3, it then drops down to zero at t = 0.3 ,and remains zero for next 0.2 seconds and one cycle is completed. The next cycle again starts at t=0.5secs. Represent the above cycle in form of a harmonic series.

Further



Undamped Free vibrations

Single degree of freedom System



This consists of a single spring attached with a single mass. The Various ways in which the equation of motion is obtained are:

a) Newton's Method b) Energy Method and c) Rayleigh Method

Newton's Method

When a mass m is attached to a spring it deflects by δ and the system is under equilibrium as mg = weight = $k\delta$, where k is the spring stiffness, defined as force per unit length. If now the mass m is given a displacement x in the downward direction and the system is allowed to vibrate, we have the following forces acting on the system: the spring force, $k(x+\delta)$ acting in the upward direction, inertia force mx'' acting in the upward direction and force mg acting in the direction of displacement x downwards. The equation of motion is written taking equilibrium of forces as:

$mx'' = - k(x+\delta) + mg$

$= -kx-k\delta+mg$

$= -kx-k\delta+k\delta$

Or $mx'' + kx = 0$, which is the governing differential equation for a single degree of freedom system.

Rewriting the equation of motion as

$x''+ (k/m) x = 0$, we have the quantity $(k/m)1/2$ as the natural frequency of the system ωn .

Energy Method:

In this method the concept of total energy of the system, which is the sum of Kinetic energy (T) and Potential energy(V) , is made use of which remains constant always for any configuration of system while it is vibrating. For a single DOF system of spring and mass, the kinetic energy is given by $(1/2)mx''$ and the potential energy stored in the system is $(1/2)kx2$. As the total energy of the system remains constant, we have $T+V = 0$ or $d(T+V)/dt = 0$.

Differentiating we have the governing differential equation as

$mx'' + kx = 0$,

and the natural frequency is given by

$\omega n = (k/m)1/2$.

Rayleigh's Method:

In this method the max kinetic energy of the system is equated to the maximum potential energy. For SHM the max. kinetic energy is at the mean position which is equated to the potential energy.

If A is the amplitude of vibration and ωn is the natural frequency the max. kinetic energy is given by $(1/2)m(\omega n\ A)2$ and max. Potential energy is $(1/2)kA2$

Equating the two equations and simplifying we have again

$\omega n = (k/m)1/2$.

UNIT – II

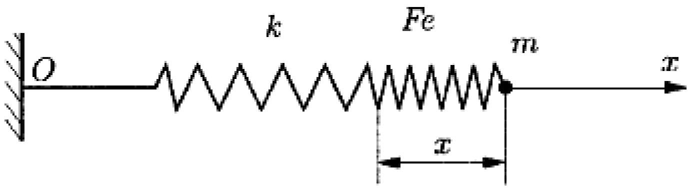
SINGLE DEGREE OF FREEDOM SYSTEMS

Free Vibrations

The mechanical model of a free undamped vibration is given in figure below and consists of a particle of mass m in a rectilinear motion. The particle is attached to one end of a mass less spring of elastic constant k.

The equation of motion on the 0x axis is

$$m\ddot{x} + kx = 0,$$



$$\ddot{x} + \frac{k}{m}x = 0.$$

If we denote k=m .o2> 0, where o is the undamped circular (angular)frequency, the differential equation of motion becomes

$$\ddot{x} + \omega^2 x = 0.$$

The characteristic equation in r is

$$r^2 + \omega^2 = 0,$$

and the general solution is

$$x = C_1 \sin \omega t + C_2 \cos \omega t,$$

Where C1 and C2 are constants and can be written as

$$x = A\sin(\omega t + \varphi),$$

$$A = \sqrt{C_1^2 + C_2^2}, \quad \tan \varphi = \frac{C_2}{C_1},$$

where A is the amplitude, and j is the phase angle.

Using the initial condition

$$t = 0 \Rightarrow \begin{cases} x = x_0 \\ \dot{x} = v_0 \end{cases}$$

Differentiating

$$\dot{x} = C_1 \omega \cos \omega t - C_2 \omega \sin \omega t,$$

The constants are

$$C_1 = \frac{v_0}{\omega}, \quad C_2 = x_0$$

$$A = \sqrt{x_0^2 + \frac{v_0^2}{\omega^2}}, \quad \varphi = \arctan \frac{x_0 \omega}{v_0}.$$

Equation of motion using this

$$x = \frac{v_0}{\omega} \sin \omega t + x_0 \cos \omega t,$$

Period of Vibration

$$T = \frac{2\pi}{\omega} = 2\pi \sqrt{\frac{m}{k}},$$

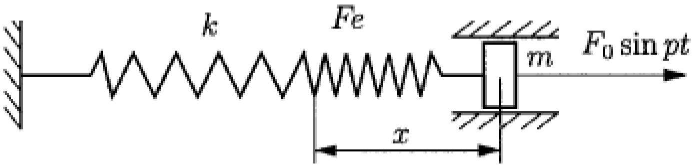
Frequency of motion is

$$f = \frac{1}{T} = \frac{\omega}{2\pi} = \frac{1}{2\pi} \sqrt{\frac{k}{m}}.$$

The undamped circular frequency of the motion is called the natural(circular or angular) frequency of the system and is given by

$$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{g}{f_{st}}}, \quad f_{st} = \frac{mg}{k},$$

Forced Vibration



An elastic system is excited by a harmonic force of the form

$$F_p = F_0 \sin pt,$$

where F0 is the amplitude of the forced vibration and p is the forced angularfrequencies. The differential equation of motion for the mechanical model in figure is

$$m\ddot{x} + kx = F_0 \sin pt,$$

$$\ddot{x} + \frac{k}{m}x = \frac{F_0}{m}\sin pt.$$

W.K.T

$$\frac{k}{m} = \omega^2; \quad \frac{F_0}{m} = q.$$

Therefore

$$\ddot{x} + \omega^2 x = q \sin pt,$$

With solution

$$x = C_1 \sin \omega t + C_2 \cos \omega t + x_p.$$

The second derivative of this equation is

$$\ddot{x}_p = -Cp^2 \sin pt.$$

The constants of these derivatives can be written as

$$C(\omega^2 - p^2) = q \Rightarrow C = \frac{q}{(\omega^2 - p^2)}.$$

Substituting this

$$x = A \sin(\omega t + \varphi) + \frac{q}{\omega^2 - p^2} \sin pt.$$

At time t = 0 displacement is 0 and hence first derivative is given as

$$\dot{x} = C_1 \omega \cos \omega t - C_2 \omega \sin \omega t + \frac{qp}{\omega^2 - p^2} \cos pt.$$

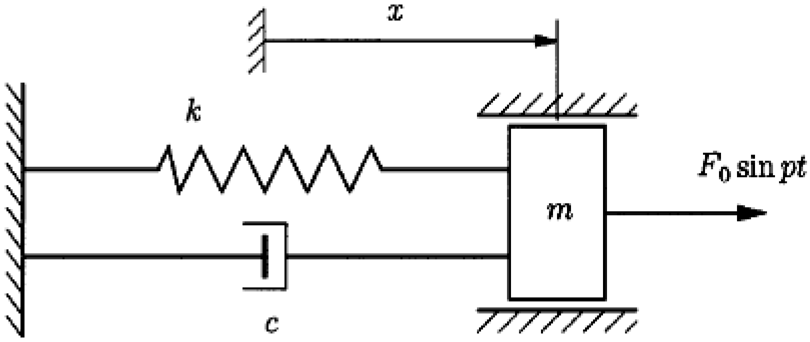
$$C_1 = -\frac{p}{\omega} \frac{q}{\omega^2 - p^2}$$

$$C_2 = 0.$$

The vibration is given by

$$x = \frac{q}{\omega^2 - p^2} \left[\sin pt - \frac{p}{\omega} \sin \omega t \right].$$

Forced damped Vibrations



Consider the mechanical system shown in the figure above, the equation of motion is given by

$$m\ddot{x} + c\dot{x} + kx = F_0 \sin pt.$$

Consider the following notation

$$\frac{c}{2m} = 2\alpha; \ \frac{k}{m} = \omega; \ \frac{F_0}{m} = q.$$

Substituting

$$\ddot{x} + 2\alpha\dot{x} + \omega^2 x = q \sin pt.$$

Case 1: ($c < c_{cr}$) or $\alpha < \omega$.

The characteristic equation is

$$r^2 + 2\alpha r + \omega^2 = 0,$$

with the roots

$$r_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega^2} = -\alpha \pm i\beta,$$

where

$$-\beta^2 = \alpha^2 - \omega^2.$$

The general solution of differential equation

$$x = x_1 + x_2,$$

Where x1 represent the solution of the differential homogenous equation, andx2 is a particular solution of the differential nonhomogeneous equation. In this case x1 represents the natural vibration of the system and x2 is the forced vibration of the system. The solution of the free damped system is

$$x_1 = e^{-\alpha t}(C_1 \sin \omega t + C_2 \cos \omega t),$$

$$x_1 = B_1 e^{-\alpha t} \sin(\omega t + \varphi),$$

$$B_1 = \sqrt{C_1^2 + C_2^2} \quad \text{and} \quad \tan \varphi = \frac{C_2}{C_1.}$$

The solution of the forced (excited) vibration is

$$x_2 = D_1 \sin pt + D_2 \cos pt.$$

$$\dot{x}_2 = D_1 p \cos pt - D_2 p \sin pt$$
$$\ddot{x}_2 = -D_1 p^2 \sin pt - D_2 p^2 \cos pt.$$

Solving this constants

$$D_1 = \frac{(\omega^2 - p^2)q}{(\omega^2 - p^2)^2 + 4\alpha^2 p^2}$$
$$D_2 = -\frac{2\alpha pq}{(\omega^2 - p^2)^2 + 4\alpha^2 p^2}.$$

Therefore

$$x_2 = \frac{(\omega^2 - p^2)q}{(\omega^2 - p^2)^2 + 4\alpha^2 p^2} \sin pt - \frac{2\alpha pq}{(\omega^2 - p^2)^2 + 4\alpha^2 p^2} \cos pt$$

$$x_2 = B_2 \sin(pt - \phi),$$

$$B_2 = \sqrt{D_1^2 + D_2^2} \quad \text{and} \quad \tan\phi = -\frac{D_2}{D_1}.$$

Therefore the motion of the system can be represented by

$$x = B_1 e^{-\alpha t} \sin(\omega t + \varphi) + B_2 \sin(pt - \phi).$$

VIBRATION MEASURING INSTRUMENTS

Strain-Gage Sensors

Many types of force and torque sensors and also motion sensors such as accelerometers are based on strain-gage measurements. Hence, strain gages are very useful in vibration instrumentation. Although strain gages measure strain, the measurements can be directly related to stress and force. Note, however, that strain gages can be used in a somewhat indirect manner (using auxiliary front-end elements) to measure other types of variables, including displacement and acceleration.

Equations for Strain-Gage Measurements

The change of electrical resistance in material when mechanically deformed is the property used in resistance-type strain gages. The resistance R of a conductor that has length L and area of cross section A is given by

$$R = \rho \frac{\ell}{A}$$

Where ϱ denotes the *resistivity* of the material. Taking the logarithm of equation

$\log R = \log \varrho + \log(l/A)$.

$$\frac{dR}{R} = \frac{d\rho}{\rho} + \frac{d(\ell/A)}{\ell/A}$$

The first term on the right-hand side of equation depends on the change in resistivity, and the second term represents deformation. It follows that the change in resistance comes from the change in shape as well as from the change in resistivity of the material. For linear deformations, the two terms on the right-hand side of equation are linear functions of strain ε; the proportionality constant of the second term, in particular, depends on Poisson's ratio of the material. Hence, the following relationship can be written for a strain-gage element:

$$\frac{\delta R}{R} = S_s \varepsilon$$

The constant S is known as the *sensitivity* or *gage factor* of the strain-gage element. The numerical values of these constant ranges from 2 to 6 for most *metallic strain-gage* elements and from 40 to200 for *semiconductor strain gages*. These two types of strain gages are discussed later. The change in resistance of a strain-gage element, which determines the associated strain is measured using a suitable electrical circuit

UNIT – III

MULTI DEGREES OF FREEDOM SYSTEMS

Modal analysis is an important tool in vibration analysis, diagnosis, design, and control. In some systems, mechanical malfunction or failure can be attributed to the excitation of their preferred motion such as modal vibrations and resonances. By modal analysis, it is possible to establish the extent and location of severe vibrations in a system. For this reason, it is an important diagnostic tool. For the same reason, modal analysis is also a useful method for predicting impending malfunctions or other mechanical problems. Structural modification and substructuring are techniques of vibration analysis and design, which are based on modal analysis. By sensitivity analysis methods using a "modal" model, it is possible to determine what degrees of freedom of a mechanical system are most sensitive to addition or removal of mass and stiffness elements. In this manner, a convenient and systematic method can be established for making structural modifications to eliminate an existing vibration problem or to verify the effects of a particular modification. A large and complex system can be divided into several subsystems that can be independently analyzed. By modal analysis techniques, the dynamic characteristics of the overall system can be determined from the subsystem information. This approach has several advantages, including: (1) subsystems can be developed by different methods such as experimentation, finite element method, or other modeling techniques and assembled to obtain the overall model; (2) the analysis of a high-order system can be reduced to several lower-order analyses; and (3) the design of a complex system can be done by designing and developing its subsystems separately. These capabilities of structural modification and substructure analysis possessed by the modal analysis method make it a useful tool in the design development process of mechanical systems. Modal control, a technique that employs modal analysis, is quite effective in the vibration control of complex mechanical systems.

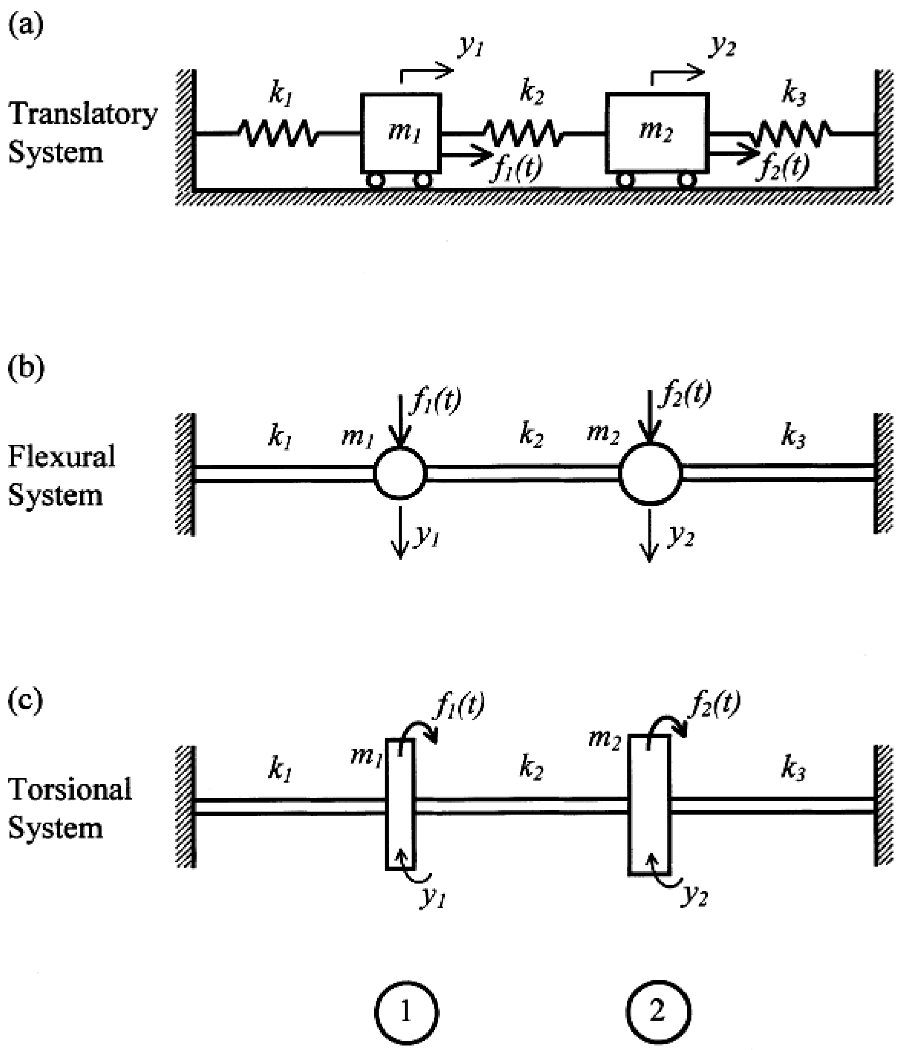
DEGREES OF FREEDOM AND INDEPENDENT COORDINATES

The geometric configuration of a vibrating system can be completely determined by a set of independent coordinates. This number of independent coordinates, for most systems, is termed the number of *degrees of freedom* (dof) of the system. For example, a particle moving freely on a plane requires two independent coordinates to completely locate it. Its motion has two degrees of freedom. A rigid body that is free to take any orientation in (the three-dimensional) space needs six independent coordinates to completely define its position. The number of degrees of freedom is equal to the number of independent "incremental" generalized coordinates that are needed to represent a general motion. In other words, it is the number of "incremental independent motions" that are possible. For *holonomic systems* (i.e., systems possessing *holonomic constraints* only), the number of independent incremental generalized coordinates is equal to the number of independent generalized coordinates; hence, either definition can be used for the number of degrees of freedom.

SYSTEM REPRESENTATION

Some damped systems do not possess real modes. If a system does not possess real modes, modal analysis could still be used but the results would be only approximately valid. In modal analysis, it is convenient to first neglect damping and develops the fundamental results, and subsequently extends them to damped systems — for example, by assuming a suitable damping model that possesses real modes. Because damping is an energy dissipation phenomenon, it is usually possible to determine a model that possesses real modes and also has an energy dissipation capacity equivalent to that of the actual system.

Consider the three undamped system representations (models) shown in the Figure below. The motion of the system (a) consists of the *translator* displacements $y1$ and $y2$ of the lumped masses $m1$ and $m2$. The masses are subjected to the external excitation forces (inputs) $f1(t)$ and $f2(t)$ and the restraining forces of the discrete, tensile-compressive stiffness (spring) elements $k1$, $k2$, and $k3$. Only two independent incremental coordinates ($\delta y1$ and $\delta y2$) are required to completely define the incremental motion of the system subject to its inherent constraints. It follows that the system has two degrees of freedom. In system (b) shown in figure the elastic stiffness to the *transverse* displacements $y1$ and $y2$ of the lumped masses is provided by three bending (*flexural*) springs, which are considered mass less. This flexural system is very much analogous to the translatory system (a) although the physical construction and the motion itself are quite different. The system (c) in Figure is the analogous *torsional* system. In this case, the lumped elements $m1$ and $m2$ should be interpreted as polar moments of inertia about the shaft axis, and $k1$, $k2$, and $k3$ as the torsional stiffness in the connecting shafts. Furthermore, the motion coordinates $y1$ and $y2$ are rotations, and the external excitations $f1(t)$ and $f2(t)$ are torques applied at the inertia elements. Practical examples where these three types of vibration system models may be useful are: (a) two-car train, (b) bridge with two separate vehicle loads, and (c) electric motor and pump combination. The three systems shown in Figure are analogous to each other in the sense that the dynamics of all three systems can be represented by similar equations of motion. For modal analysis, it is convenient to express the system equations as a set of coupled second-order differential equations in terms of the displacement variables (coordinates) of the inertia elements.



Since in modal analysis one is concerned with *linear* systems, the system parameters can be given by a *mass matrix* and a *stiffness matrix* or a *flexibility matrix*. Lagrange's equations of motion directly yield these matrices. An intuitive method for identifying the stiffness and mass matrices is presented below. The linear, lumped-parameter, un-damped systems shown in figure satisfy the set of dynamic equations

$$\begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} \begin{bmatrix} \ddot{y}_1 \\ \ddot{y}_2 \end{bmatrix} + \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}$$

or

$$\boldsymbol{M}\ddot{\boldsymbol{y}} = \boldsymbol{K}\boldsymbol{y} = \boldsymbol{f}$$

Here, ***M*** is the inertia matrix, which is the generalized case of mass matrix, and ***K*** is the stiffness matrix. There are many ways to derive equations. Below is described an approach, termed the *influence coefficient method*, that accomplishes the task by separately determining ***K*** and ***M***.

STIFFNESS AND FLEXIBILITY MATRICES

In the systems shown in Figure, suppose the accelerations 1 and 2 both are zero at a particular instant, so that the inertia effects are absent. The stiffness matrix ***K*** is given under these circumstances, by the *constitutive relation* for the spring elements. The elements of the stiffness matrix, in this two-degree-of-freedom (2 d of) case, are explicitly given by

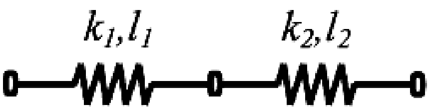
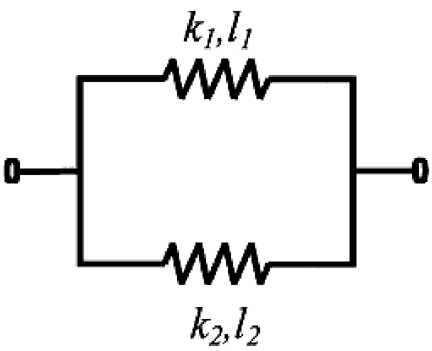
$$\begin{bmatrix} f_1 \\ f_2 \end{bmatrix} = \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$$\boldsymbol{f} = \boldsymbol{K}\boldsymbol{y}$$

Where

$$\boldsymbol{K} = \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix}$$

Combination Rules for Stiffness and Flexibility Elements

Connection	Graphical Representation	Combined Stiffness	Combined Flexibilit
Series		$\frac{1}{(1/k_1+1/k_2)}$	$l_1 + l_2$
Parallel		$k_1 + k_2$	$\frac{1}{(1/l_1+1/l_2)}$

Lagrangian Mechanics

A vibrating system can be interpreted as a collection of mass particles. In the case of distributed systems, the number of particles is infinite. The flexibility and damping effects can be introduced as forces acting on these particles. It follows that Newton's second law for a mass particle forms the basis of describing vibratory motions. System equations can be obtained directly by applying Newton's second law to each particle. It is convenient, however, to use Lagrange's equations for this purpose, particularly when the system is relatively complex. A variational principle known as Hamilton's principle, which can be established from Newton's second law, is the starting point in the derivation of Lagrange's equations.

Consider such single particle

Kinetic energy $$T = \int \mathbf{v} \cdot d\mathbf{p}$$

Kinetic coenergy $$T^* = \int \mathbf{p} \cdot d\mathbf{v}$$

Note: $T + T^* = \mathbf{v} \cdot \mathbf{p}$

In classical mechanics, the *constitutive relation* between velocity v and linear momentum $\boldsymbol{p}$ is *linear*,

$p = mv$

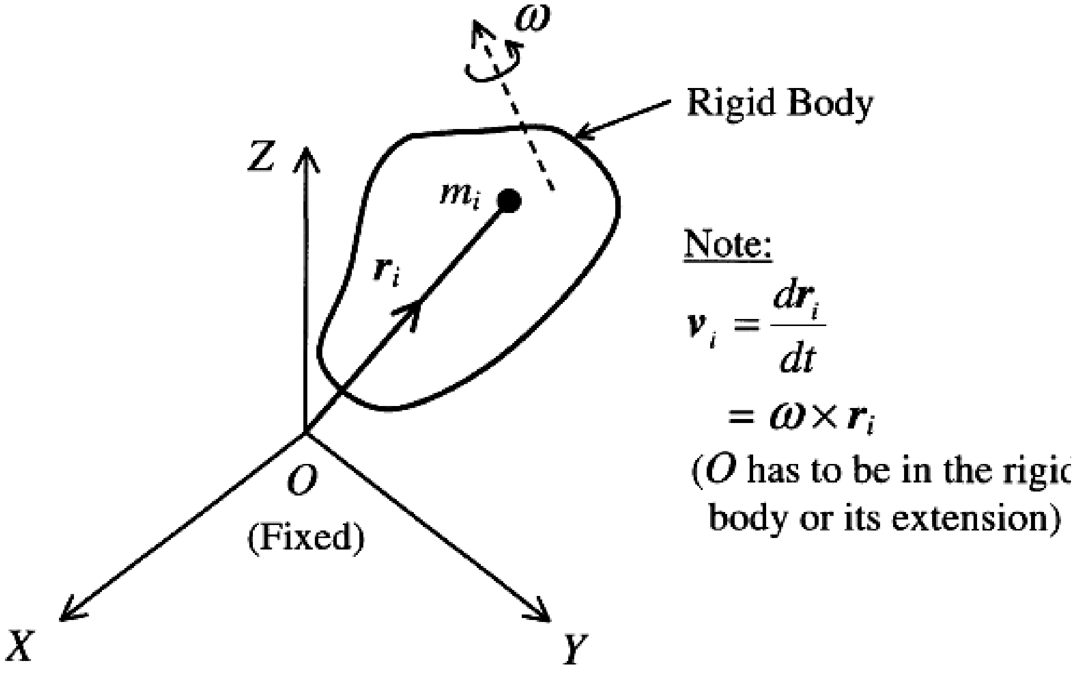
$$T = T^* = \frac{1}{2} m\boldsymbol{v} \cdot \boldsymbol{v} = \frac{1}{2} mv^2 \qquad \text{(for a particle)}$$

For a system of particles

$$T^* = \frac{1}{2} \sum m_i \boldsymbol{v}_i \cdot \boldsymbol{v}_i$$

For a rotating rigid body

$$T^* = \frac{1}{2} \sum m_i (\boldsymbol{\omega} \times r_i) \cdot (\boldsymbol{\omega} \times r_i)$$



$$T^* = \frac{1}{2}\underbrace{\sum m_i\left[r_i \times (\boldsymbol{\omega} \times r_i)\right]}_{H_o} \cdot \boldsymbol{\omega}$$

$$T^* = \frac{1}{2}\boldsymbol{\omega}^T H_o = \frac{1}{2}\boldsymbol{\omega}^T [I]_o \boldsymbol{\omega}$$

This equation can be modified for a centroid C and can be given as

$$T^* = \frac{1}{2}Mv_c^2 + \frac{1}{2}\boldsymbol{\omega}^T [I]_c \boldsymbol{\omega}$$

Hamilton's Principle

For a holonomic system, the Lagrangian L is given by

$$L = T^* - V$$

Consider the variation integral

$$\delta H = \int_{t_0}^{t_f}\left[\delta L + \sum_{j=1}^{n} Q_j \delta q_j\right]dt$$

In which *Qj* are the *nonconservative* generalized forces corresponding to the generalized coordinates *qj*. For a motion trajectory, *t*0 and *tf* are the initial and the final times, respectively. Hamilton's principle states that this trajectory corresponds to a *natural motion* of the system if and only if δ*H*= 0 for arbitrary δ*qj* about the trajectory.

Lagrange's Equation

Note that *L* is a function of *qj* and q_i in general, because *V* is a function of *qj* and q_i*T** is a function of *qj* and q_i . Hence,

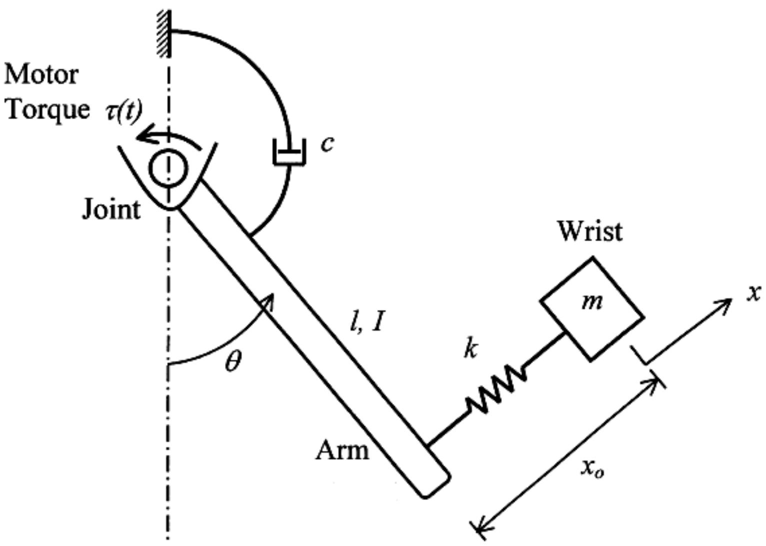
$$L = L(q_1, q_2, \ldots, q_n, \dot{q}_1, \dot{q}_2, \ldots, \dot{q}_n)$$

Therefore from Hamilton's equation

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right)-\frac{\partial L}{\partial q_j}=Q_j \qquad j=1,\ 2,\ \ldots,\ n$$

These are termed Lagrange's equations, and represent a complete set of equations of motion.

Figure below shows a simplified model that can be used to study the mechanical vibrations that are excited by the control-loop disturbances in a single-link robot arm. The length of the arm is *l*, the mass is *M*, and the moment of inertia about the joint is *I*. The gripper hand (end effector) is modelled as a mass *m* connected to the arm through a spring of stiffness *k*. The joint has an effective viscous damping constant *c* for rotary motions. The motor torque (applied at the joint) is$\tau(t)$.



This is a two-degree-of-freedom holonomic system. The angle of rotation θ of the arm and spring deflection x from the unstretched position are chosen as the generalized coordinates.

Keeping x fixed, increment θ by $\delta\theta$. The corresponding incremental work due to *non-conservativeforces* is

$$\delta W_\theta = \tau(t)\delta\theta - c\dot{\theta}\delta\theta.$$

Note that the damping torque c acts opposite to the increment $\delta\theta$. Thus, the generalized force is given by

$$F_\theta = \tau(t) - c\dot{\theta}$$

Keeping θ fixed, increment x by δx. The corresponding incremental work due to non-conservative forces is $\delta Wx = 0$.

Hence, $Fx = 0$.

Lagrangian

The total kinetic coenergy (= kinetic energy in these Newtonian systems) is

$$T^* = \frac{1}{2} I\dot{\theta}^2 + \frac{1}{2} m\left(l\dot{\theta} + \dot{x}\right)^2$$

Note: $\left(l\dot{\theta} + \dot{x}\right)^2$ is not exact (there is a nonlinear term that is neglected).
The potential energy (due to gravity and spring) is given by:

$$V = Mg\left(-\frac{1}{2} l\cos\theta\right) + mg\left(-l\cos\theta + \left(x + x_0\right)\sin\theta\right) + \frac{1}{2} kx^2$$

Note that the centroid of the arm is assumed to be halfway along the link. It follows that the Lagrangian is given by:

$$L = \frac{1}{2} I\dot{\theta}^2 + \frac{1}{2} m\left(l\dot{\theta} + \dot{x}\right)^2 + \left(\frac{M}{2} + m\right)gl\cos\theta - mg\left(x + x_0\right)\sin\theta - \frac{1}{2} kx^2$$

$$\text{For } \theta: \frac{d}{dt}\left[I\dot{\theta} + m\left(l\dot{\theta} + \dot{x}\right)l\right] - \left[-\left(\frac{M}{2} + m\right)gl\sin\theta - mg\left(x + x_0\right)\cos\theta\right] = \tau(1) - c\dot{\theta}$$

$$\text{For } x: \frac{d}{dt}\left[m\left(l\dot{\theta} + \dot{x}\right)\right] - \left[mg\sin\theta - kx\right] = 0$$

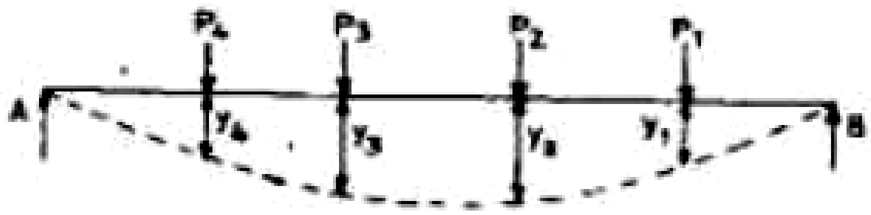
UNIT – IV

APPROXIMATE METHODS

Rayleigh's Method

Rayleigh's method is used to find natural frequency of the system when transverse point loads are acting on the beam or shaft. Good estimate of fundamental frequency can be made by assuming the suitable deflection curve for the fundamental mode. The maximum kinetic energy is equated to maximum potential energy of the system to determine the natural frequency.

Let us consider the shaft as shown in the figure below. Several point loads P_1, P_2, P_3, P_4 etc. are acting transversely, suppose y_1, y2, y3, y4 etc be maximum deflections under the influence of point load



The maximum potential energy of the system can be written as

$$P.E. = \frac{1}{2}P_1y_1 + \frac{1}{2}P_2y_2 + \frac{1}{2}P_3y_3 + \frac{1}{2}P_4y_4$$

$$P.E. = \frac{1}{2}\sum Py \quad ...(6.8.1)$$

The maximum kinetic energy of the system can be written as

$$K.E. = \frac{1}{2g}P_1(\omega y_1)^2 + \frac{1}{2g}P_2(\omega y_2)^2 + \frac{1}{2g}P_3(\omega y_3)^2 + \frac{1}{2g}P_4(\omega y_4)^2$$

$$= \frac{\omega^2}{2g}\Sigma Py^2 \quad ...(6.8.2)$$

where ω = natural frequency of vibration.

Equating the maximum kinetic energy to maximum potential energy, we have

$$\frac{\omega^2}{2g}\Sigma Py^2 = \frac{1}{2}\Sigma Py$$

$$\omega = \sqrt{g\frac{\Sigma Py}{\Sigma Py^2}} \qquad ...(6.8.3)$$

The above equation can be written in a more generalised way by including the distributed mass of the beams. If m is the mass of the beam per unit length and y is the assumed deflection curve, the maximum potential energy of beam of length l is expressed as

$$\text{P.E.} = \frac{1}{2}\int_0^l M\,d\theta \qquad ...(6.8.4)$$

where M = bending moment

$d\,\theta$ = change in slope over a distance dx

From beam theory, we know that

$\frac{M}{I} = \frac{E}{R}$ where R is the radius of curvature and EI is the flexural rigidity.

Also $$\frac{1}{R} = \frac{d\theta}{dx} = \frac{d^2y}{dx^2}$$

Thus $$M = \frac{EI}{R} = EI\frac{d^2y}{dx^2} \qquad ...(6.8.5)$$

and $$\frac{d\theta}{dx} = \frac{d^2y}{dx^2}, \text{ or } d\theta = \left(\frac{d^2y}{dx^2}\right)dx \qquad ...(6.8.6)$$

Potential Energy can be written as

$$P.E. = \frac{1}{2}\int_0^l EI\left(\frac{d^2y}{dx^2}\right)^2 dx$$

Kinetic Energy can be written as

$$K.E. = \frac{1}{2}\int_0^l m(\omega y)^2\, dx$$

Equating

$$\frac{1}{2}\int_0^l m(\omega y)^2\, dx = \frac{1}{2}\int_0^l EI\left(\frac{d^2y}{dx^2}\right)^2 dx$$

$$\omega^2 = \frac{\int_0^l EI\left(\frac{d^2y}{dx^2}\right)^2 dx}{\int_0^l m y^2 dx}$$

$$= \frac{EI}{m}\frac{\int_0^l \left(\frac{d^2y}{dx^2}\right)^2 dx}{\int_0^l y^2 dx}$$

This can be used to find the natural frequencies of the oscillating medium.

Holzer's Method

Holzer's method is a trial and error method used to find the natural frequency and mode shape of multi-mass lumped parameter system. This can be applied to both free and forced vibrations. This method can be used for the analysis of damped, un-damped, semi definite systems with fixed ends having linear and angular motions. At first a trial frequency of the system is assumed, a solution is found when the trial frequency satisfies the constraints of the system.

Example

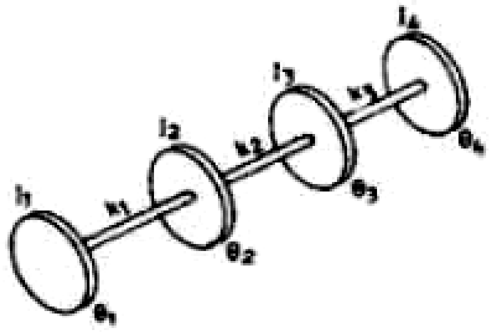


Fig. 6.7.

$$I_1\ddot{\theta}_1 + k_1(\theta_1 - \theta_2) = 0 \quad ...(6.9.1)$$

$$I_2\ddot{\theta}_2 + k_1(\theta_2 - \theta_1) + k_2(\theta_2 - \theta_3) = 0 \quad ...(6.9.2)$$

$$I_3\ddot{\theta}_3 + k_2(\theta_3 - \theta_2) + k_3(\theta_3 - \theta_4) = 0 \quad ...(6.9.3)$$

$$I_4\ddot{\theta}_4 + k_3(\theta_4 - \theta_3) = 0 \quad ...(6.9.4)$$

The motions are harmonic at a principal mode of vibration. Assuming $\theta_i = \phi_i \cdot \sin \omega t$ and substituting it in the above equations, we get

$$\omega^2 I_1 \phi_1 = k_1(\phi_1 - \phi_2) \quad ...(6.9.5)$$

$$\omega^2 I_2 \phi_2 = k_1(\phi_2 - \phi_1) + k_2(\phi_2 - \phi_3) \quad ...(6.9.6)$$

$$\omega^2 I_3 \phi_3 = k_2(\phi_3 - \phi_2) + k_3(\phi_3 - \phi_4) \quad ...(6.9.7)$$

$$\omega^2 I_4 \phi_4 = k_3(\phi_4 - \phi_3) \quad ...(6.9.8)$$

Summing the various terms of the above equations, we get

$$\sum_{i=1}^{4} \omega^2 I_i \phi_i = 0 \quad ...(6.9.9)$$

For a set of n discs equation (6.9.9) can be written as

$$\sum_{i=1}^{n} \omega^2 I_i \phi_i = 0 \quad ...(6.9.10)$$

In the above equation it is explained that the sum of the inertia orques $k_1(\phi_1 - \phi_2)$, $k_2(\phi_3 - \phi_2)$, etc., must be zero and the assumed trial requency ω must satisfy this equation.

Procedure

1. Assume a trial frequency ω
2. Take ϕ_1 as unity arbitrarily.
3. Calculate ϕ_2 from equation (6.9.5) as

$$\phi_2 = \phi_1 - \frac{I_1 \omega^2 \phi_1}{k_i} \quad ...(6.9.11)$$

or $$= \left(1 - \frac{I_1 \omega^2}{k_1}\right)\phi_1$$

Similarly, ϕ_3 and ϕ_4 can be computed from equations (6.9.6) and (6.9.7) as

$$\phi_3 = \phi_2 - \frac{\omega^2(I_1\phi_1 + I_2\phi_2)}{k_2} \quad ...(6.9.12)$$

$$\phi_4 = \phi_3 - \frac{(I_1\phi_1 + I_2\phi_2 + I_3\phi_3)\,\omega^2}{k_3} \quad ...(6.9.13)$$

4. The values of ϕ_1, ϕ_2, ϕ_3 and ϕ_4 are put in equation (6.9.9) to check whether the equation is satisfied or not. If equation (6.9.9) is not satisfied, a new trial value of ω is assumed and the whole process is repeated.

5. Prepare a table showing various terms as :

Row	I	ϕ	$I\phi\omega^2$	$\Sigma I\phi\omega^2$	k	$\frac{1}{k}\Sigma I\phi\omega^2$
1.	I_1	ϕ_1	$I_1\phi_1\omega^2$		k_1	
2.	I_2	ϕ_2	$I_2\phi_2\omega^2$	Summation	k_2	Summation
3.	I_3	ϕ_3	$I_3\phi_3\omega^2$		k_3	

and so on.

UNIT – V

ELEMENTS OF AEROELASTICITY

Aircraft structures, being extremely flexible, are prone to distortion under load. When these loads are caused by aerodynamic forces, which themselves depend on the geometry of the structure and the orientation of the various structural components to the surrounding airflow, then structural distortion results in changes in aerodynamic load, leading to further distortion and so on. The interaction of aerodynamic and elastic forces is known as *aeroelasticity*.

Two distinct types of aeroelastic problem occur. One involves the interaction of aerodynamic and elastic forces of the type described above. Such interactions may exhibit divergent tendencies in a too flexible structure, leading to failure, or, in an adequately stiff structure, converge until a condition of stable equilibrium is reached. In this type of problem *static* or *steady state* systems of aerodynamic and elastic forces produce such aeroelastic phenomena as *divergence* and *control reversal.* The second class of problem involves the inertia of the structure as well as aerodynamic and elastic forces. Dynamic loading systems, of which gusts are of primary importance, induce oscillations of structural components. If the natural or resonant frequency of the component is in the region of the frequency of the applied loads then the amplitude of the oscillations may diverge, causing failure.

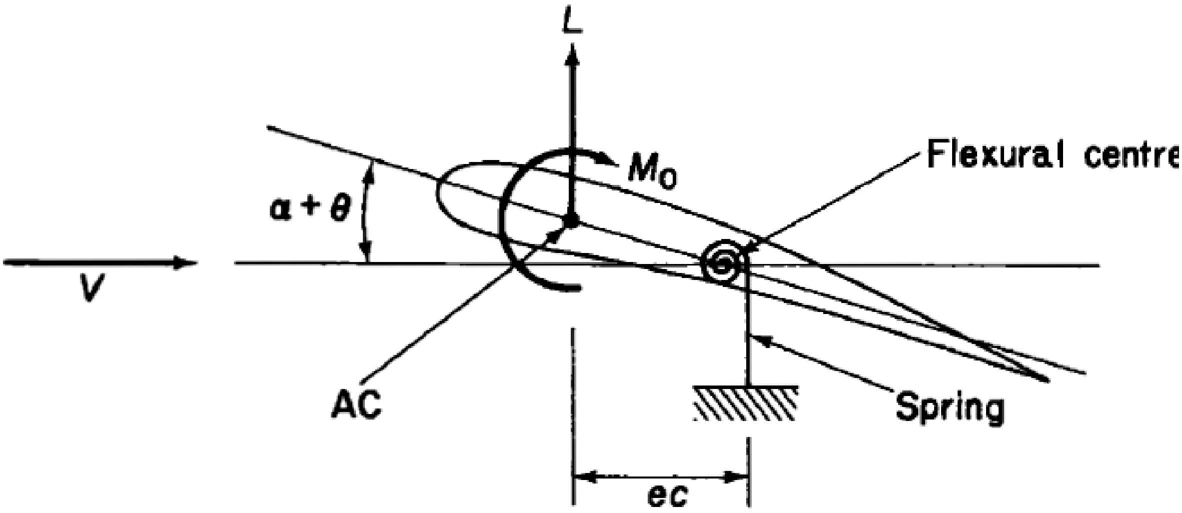
Wing torsional divergence (two-dimensional case)

The most common divergence problem is the torsional divergence of a wing. It is useful, initially, to consider the case of a wing of area ***S*** without ailerons and in a two-dimensional flow, as shown in figure below**.** The torsional stiffness of the wing, which we shall represent by a spring of stiffness ***K,*** resists the moment of the lift vector, ***L,*** and the wing pitching moment *Mo,* acting at the aerodynamic centre of the wing section.

For moment equilibrium of the wing section about the aerodynamic centre we have

$$M_0 + Lec = K\theta$$

Where **ec** is the distance of the aerodynamic centre forward of the flexural centre expressed in terms of the wing chord, ***c,*** and theta is the elastic twist of the wing.



From aerodynamic theory

$$M_0 = \tfrac{1}{2}\rho V^2 S c C_{M,0}, \quad L = \tfrac{1}{2}\rho V^2 S C_L$$

$$\tfrac{1}{2}\rho V^2 S (c C_{M,0} + ec C_L) = K\theta$$

$$C_L = C_{L,0} + \frac{\partial C_L}{\partial \alpha}(\alpha + \theta)$$

In which (Y is the initial wing incidence or, in other words, the incidence corresponding to given flight conditions assuming that the wing is rigid and CL. O is the wing lift coefficient at zero incidence, then

$$\frac{1}{2}\rho V^2 S \left[c C_{M,0} + e C_{L,0} + ec \frac{\partial C_L}{\partial \alpha}(\alpha + \theta) \right] = K\theta$$

Rearranging

$$\theta\left(K - \frac{1}{2}\rho V^2 Sec\frac{\partial C_{\mathrm{L}}}{\partial \alpha}\right) = \frac{1}{2}\rho V^2 Sc\left(C_{\mathrm{M,0}} + eC_{\mathrm{L,0}} + e\frac{\partial C_{\mathrm{L}}}{\partial \alpha}\alpha\right)$$

$$\theta = \frac{\frac{1}{2}\rho V^2 Sc[C_{\mathrm{M,0}} + eC_{\mathrm{L,0}} + e(\partial C_{\mathrm{L}}/\partial \alpha)\alpha]}{K - \frac{1}{2}\rho V^2 Sec(\partial C_{\mathrm{L}}/\partial \alpha)}$$

Shows that divergence occurs (Le. ***6*** becomes infinite) when

$$K = \frac{1}{2}\rho V^2 Sec\frac{\partial C_{\mathrm{L}}}{\partial \alpha}$$

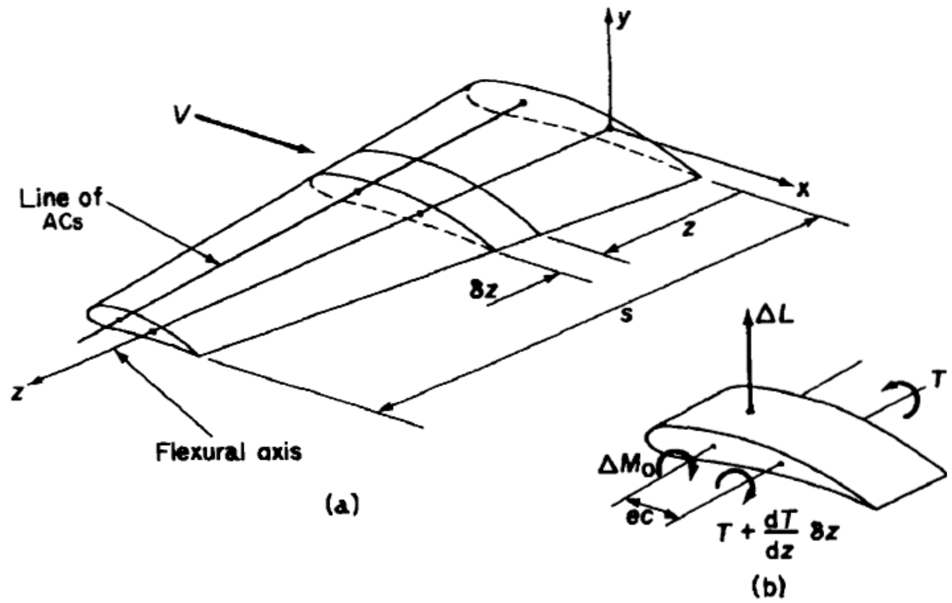
The divergence speed v_d is then

$$V_{\mathrm{d}} = \sqrt{\frac{2K}{\rho Sec(\partial C_{\mathrm{L}}/\partial \alpha)}}$$

We see from the equation that v_d may be increased either by stiffening the wing (increasing ***K)*** or by reducing the distance **ec** between the aerodynamic and flexural centres. The former approach involves weight and cost penalties so that designers usually prefer to design a wing structure with the flexural centre as far forward as possible. If the aerodynamic centre coincides with or is aft of the flexural centre then the wing is stable at all speeds.

Wingtorsional divergence (finite wing)

We shall consider the simple case of a straight wing having its flexural axis nearly perpendicular to the aircraft's plane of symmetry. We shall also assume that wing cross-sections remain undistorted under the loading. Applying strip theory in the usual manner, that is we regard a small element of chord c and spanwise width 6z as acting independently of the remainder of the wing and consider its equilibrium, we have from figure given below neglecting wing weight.



$$\left(T + \frac{\mathrm{d}T}{\mathrm{d}z}\delta z\right) - T + \Delta Lec + \Delta M_0 = 0$$

Where *T* is the applied torque at any spanwise section *z* and *AL* and *AMo* are the lift and pitching moment on the elemental strip acting at its aerodynamic centre. As *dz* approaches zero

$$\frac{\mathrm{d}T}{\mathrm{d}z} + ec\frac{\mathrm{d}L}{\mathrm{d}z} + \frac{\mathrm{d}M_0}{\mathrm{d}z} = 0$$

Then

$$\Delta L = \frac{1}{2}\rho V^2 c\delta z\frac{\partial c_1}{\delta\alpha}(\alpha + \theta)$$

Where **dcl** /*acu*is the local two-dimensional lift curve slope and

$$\Delta M_0 = \tfrac{1}{2}\rho V^2 c^2 \delta z c_{\mathbf{m},0}$$

In which $\mathbf{c}_{\mathbf{m},0}$is the local pitching moment coefficient about the aerodynamic centre.

From torsion theory

$$\frac{d^2\theta}{dz^2} + \frac{\frac{1}{2}\rho V^2 ec^2(\partial c_1/\partial\alpha)\theta}{GJ} = \frac{-\frac{1}{2}\rho V^2 ec^2(\partial c_1/\partial\alpha)\alpha}{GJ} - \frac{\frac{1}{2}\rho V^2 c^2 c_{m,0}}{GJ}$$

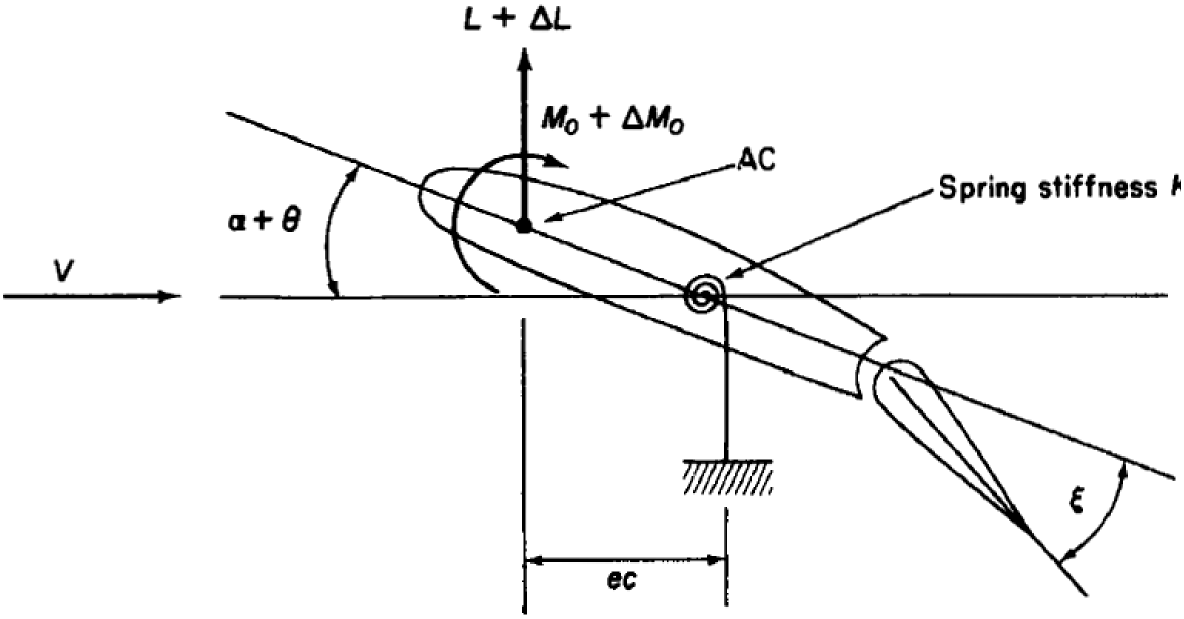
By using second order differential equations

$$\theta = \left[\frac{c_{m,0}}{e(\partial c_1/\partial\alpha)} + \alpha\right]\left[\frac{\cos\lambda(s-z)}{\cos\lambda s} - 1\right]$$

Aileron effectiveness and reversal

The flexibility of the major aerodynamic surfaces (wings, vertical and horizontal tails) adversely affects the effectiveness of the corresponding control surfaces (ailerons, rudder and elevators). For example, the downward deflection of an aileron causes a nose down twisting of the wing which consequently reduces the aileron incidence. Thus, the wing twist tends to reduce the increase in lift produced by the aileron deflection, And thereby the rolling moment to a value less than that for a rigid wing. The aerodynamic twisting moment on the wing due to aileron deflection increases as the square of the speed but the elastic restoring moment is constant since it depends on the torsional stiffness of the wing structure. Therefore, ailerons become markedly less effective as the speed increases until, at a particular speed, the ***aileron reversal speed,*** aileron deflection does not produce any rolling moment at all. At higher speeds reversed aileron movements are necessary in that a positive increment of wing lift requires an upward aileron deflection and vice versa. Similar, less critical, problems arise in the loss of effectiveness and reversal of the rudder and elevator controls. They are complicated by the additional deformations of the fuselage and tailplane-fuselage attachment points, which may be as important as the deformations of the tailplane itself.

We shall illustrate the problem by investigating**,** the case of a wingaileron combination in a two-dimensional flow. In figure below an aileron deflection ∑produces *changes* ***ΔL*** and ***ΔM,*** in the wing lift, ***L,*** and wing pitching moment M_0;



These in turn cause an elastic twist, *theta,* of the wing. Thus

$$\Delta L = \left(\frac{\partial C_L}{\partial \alpha}\theta + \frac{\partial C_L}{\partial \xi}\xi\right)\frac{1}{2}\rho V^2 S$$

Where *dCL/da* has been previously defined and **dCL/da(**is the rate of change of lift coefficient with aileron angle. Also

$$\Delta M_0 = \frac{\partial C_{M,0}}{\partial \xi}\xi\frac{1}{2}\rho V^2 Sc$$

In which *dCM,o/dt* is the rate of change of wing pitching moment coefficient with aileron deflection. The moment produced by these increments in lift and pitching moment is equilibrated by an increment of torque *AT* about the flexural axis. Hence

$$\Delta T = K\theta = \frac{1}{2}\rho V^2 Sc\left[\left(\frac{\partial C_L}{\partial \alpha}\theta + \frac{\partial C_L}{\partial \xi}\xi\right)e + \frac{\partial C_{M,0}}{\partial \xi}\xi\right]$$

Isolating *theta* from

$$\theta = \frac{\frac{1}{2}\rho V^2 Sc[(\partial C_L/\partial \xi)e + \partial C_{M,0}/\partial \xi]\xi}{K - \frac{1}{2}\rho V^2 Sce(\partial C_L/\partial \alpha)}$$

Substituting

$$\Delta L = \frac{1}{2}\rho V^2 S\left[\frac{[\frac{1}{2}\rho V^2 Sc(\partial C_{M,0}/\partial \xi)(\partial C_L/\partial \alpha) + K(\partial C_L/\partial \xi)]}{K - \frac{1}{2}\rho V^2 Sce(\partial C_L/\partial \alpha)}\right]\xi$$

The increment of wing lift is therefore a linear function of aileron deflection and becomes zero, that is aileron reversal occurs, when

$$\frac{1}{2}\rho V^2 Sc\frac{\partial C_{M,0}}{\partial \xi}\frac{\partial C_L}{\partial \alpha} + K\frac{\partial C_L}{\partial \xi} = 0$$

Hence the aileron reversal speed, *Vr*

$$V_r = \sqrt{\frac{-K(\partial C_L/\partial \xi)}{\frac{1}{2}\rho Sc(\partial C_{M,0}/\partial \xi)(\partial C_L/\partial \alpha)}}$$

We may define aileron effectiveness at speeds below the reversal speed in terms of aileron effectiveness = **ΔL/ΔLR** the lift **ΔLR** produced by an aileron deflection on a rigid wing. Thus

$$\Delta L_R = \frac{\partial C_L}{\partial \xi}\xi\frac{1}{2}\rho V^2 S$$

Substituting

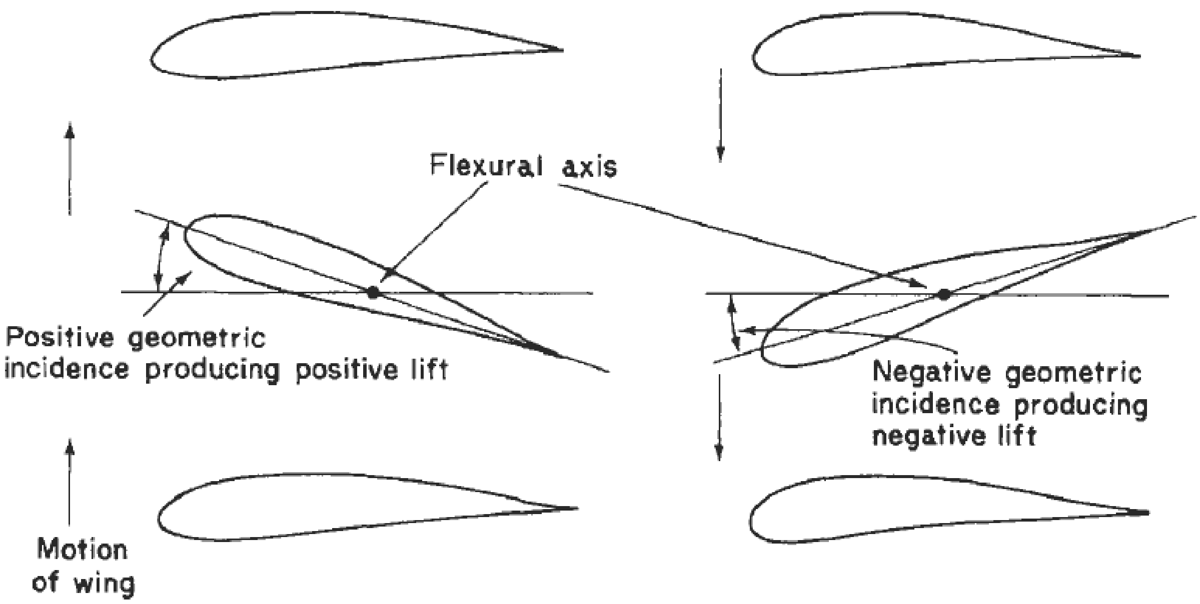
$$\text{aileron effectiveness} = \frac{\frac{1}{2}\rho V^2 Sc(\partial C_{M,0}/\partial \xi)(\partial C_L/\partial \alpha) + K(\partial C_L/\partial \xi)}{[K - \frac{1}{2}\rho V^2 Sce(\partial C_L/\partial \alpha)]\partial C_L/\partial \xi}$$

This can be expressed in wing divergence speed and aileron reversal speed

$$\text{aileron effectiveness} = \frac{1 - V^2/V_r^2}{1 - V^2/V_d^2}$$

Wing Flutter

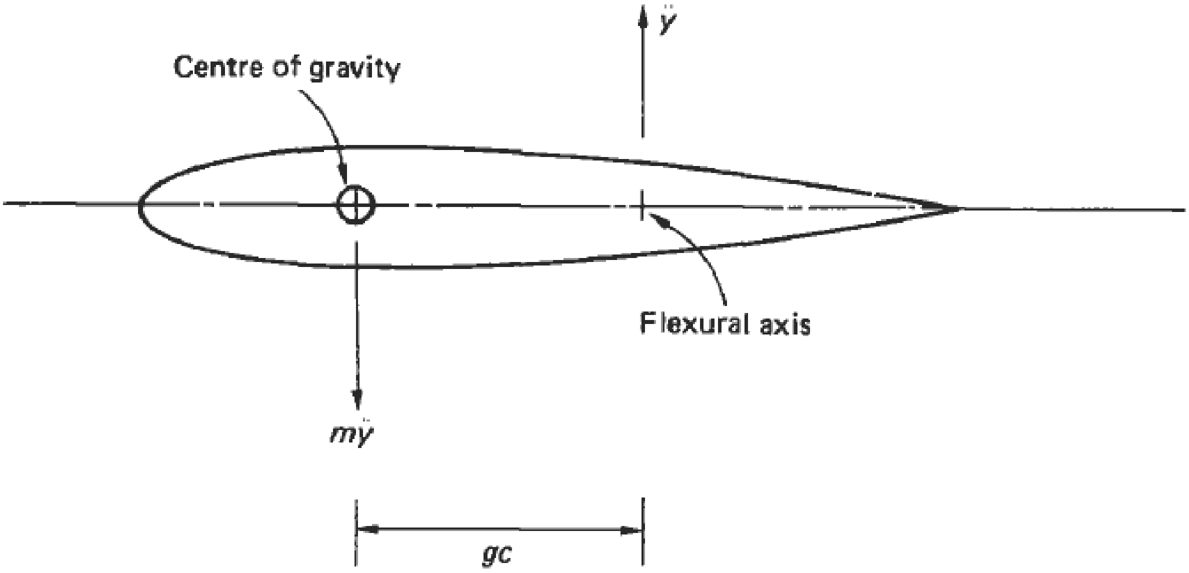
Flutter has been defined as the dynamic instability of an elastic body in an airstream and is produced by aerodynamic forces which result from the deflection of the elastic body from its underformed state. The determination of critical or flutter speeds for the continuous structure of an aircraft is a complex process since such a structure possesses an infinite number of ***natural*** or ***normal*** modes of vibration. Simplifying assumptions, such as breaking down the structure into a number of concentrated masses connected by weightless elastic beams ***(lumped mass concept)*** are made, but whatever method is employed the natural modes and frequencies of vibration of the structure must be known before flutter speeds and frequencies can be found. It is found most frequently in aircraft structures subjected to large aerodynamic loads such as wings, tail units and control surfaces. Flutter occurs at a critical or flutters speed **Vf** which in turn is defined as the lowest airspeed at which a given structure will oscillate with sustained simple harmonic motion. Flight at speeds below and above the flutter speed represents conditions of stable and unstable (that is divergent) structural oscillation respectively. Generally, an elastic system having just one degree of freedom cannot be unstable unless some peculiar mechanical characteristic exists such as a negative spring force or a negative damping force. However, it is possible for systems with two or more degrees of freedom to be unstable without possessing unusual characteristics. The forces associated with each individual degree of freedom can interact, causing divergent oscillations for certain phase differences. The flutter of a wing in which the flexural and torsional modes are coupled **is** an important example of this type of instability. Some indication of the physical nature of ***wing bending-torsion flutter*** may be had from an examination of aerodynamic and inertia forces during a combined bending and torsional oscillation in which the individual motions are 90' out of phase. In a pure bending or pure torsional oscillation the aerodynamic forces produced by the effective wing incidence oppose the motion; the geometric incidence in pure bending remains constant and therefore does not affect the aerodynamic damping force, while in pure torsion the geometric incidence produces aerodynamic forces which oppose the motion during one half of the cycle but assist it during the other half so that the overall effect is nil. Thus, pure bending or pure torsional oscillations are quickly damped out. This is not the case in the combined oscillation when the maximum twist occurs at zero bending and vice versa; that is a 90" phase difference. Consider the wing shown in figure shown below in various stages of a bending-torsion oscillation. At the position of zero bending the twisting of the wing causes a positive geometric incidence and therefore an aerodynamic force in the same direction as the motion of the wing. A similar but reversed situation exists as the wing moves in a downward direction; the negative geometric incidence due to wing twist causes a downward aerodynamic force. It follows that, although the effective wing incidence produces aerodynamic forces which oppose the motion at all stages, the aerodynamic forces associated with the geometric incidence have a destabilizing effect. At a certain speed - the flutter speed *V,* - this destabilization action becomes greater than the stabilizing forces and the oscillations diverge. In practical cases the bending and torsional oscillations would not be as much as 90^{o} out of phase; however, the same basic principles apply. The type of flutter described above, in which two distinctly different types of oscillating motion interact such that the resultant motion is divergent, is known as ***classical flutter.*** Other types of flutter, ***non-classical flutter,*** may involve only one type of motion. For example, ***stallingflutter*** of a wing occurs at a high incidence where, for particular positions of the spanwise axis of twist, self-excited twisting oscillations occur which, above a critical speed, diverge. Another non-classical form of flutter, ***aileron buzz,*** occurs at high subsonic speeds and is associated with the shock wave on the wing forward of the aileron. If the aileron oscillates downwards the flow over the upper surface of the wing accelerates, intensifying the shock and resulting in a reduction in pressure in the boundary layer behind the shock. The aileron, therefore, tends to be sucked back to its neutral position. When the aileron rises the shock intensity reduces and the pressure in the boundary layer increases, tending to push the aileron back to its neutral position. At low frequencies these pressure changes are approximately 180" out of phase with the aileron deflection and therefore become aerodynamic damping forces. At higher frequencies a component of pressure appears in phase with the aileron velocity which excites the oscillation. If this is greater than all other damping actions on the aileron a high frequency oscillation results in which only one type of motion, rotation of the aileron about its hinge, is present, i.e. aileron buzz. Aileron buzz may be prevented by employing control jacks of sufficient stiffness to ensure that the natural frequency of aileron rotation is high. ***Bufeting*** is produced most commonly in a tail plane by eddies caused by poor airflow in the wing wake striking the tail plane at a frequency equal to its natural frequency; a resonant oscillation having one degree of freedom could then occur. The problem may be alleviated by proper positioning of the tail plane and clean aerodynamic design.



Coupling

We have seen that the classical flutter of an aircraft wing involves the interaction offlexural and torsional motions. Separately neither motion will cause flutter but together, at critical values of amplitude and phase angle, the forces produced by one motion excite the other; the two types of motion are then said to be ***coupled.***

Various forms of coupling occur: inertial, aerodynamic and elastic. The cross-section of a small length of wing is shown below. Its centre of gravity **is** a distance ***gc*** ahead of its flexural **axis, *c*** is the wing section chord and the mass of the **small** length of wing is ***m.*** If the length of wing is subjected to an upward acceleration ***j ;***an accompanying inertia force ***my*** acts at its centre of gravity in a downward direction, thereby producing a nose down torque about the flexural axis of ***mygc,*** causing the wing to twist. The vertical motion therefore induces a twisting motion by virtue of the inertia forces present, i.e. ***inertial coupling.*** Conversely, an angular acceleration ***ti*** about the flexural axis causes a linear acceleration of ***gc&*** at the centre of gravity with a corresponding inertia force of ***mgcii.*** Thus, angular acceleration generates a force producing translation, again inertial coupling. Note that the inertia torque due to unit linear acceleration***(mgc)*** is equal to the inertia force due to unit angular acceleration ***(mgc);*** the inertial coupling therefore possesses symmetry.



Aerodynamic coupling is associated with changes of lift produced by wing rotation or translation. **A** change of wing incidence, that is a rotation of the wing, induces a change of lift which causes translation while a translation of velocity ***3,*** say, results in an effective change in incidence, thereby yielding a lift which causes rotation. These aerodynamic forces, which oscillate in a flutter condition, act through a centre analogous to the aerodynamic centre of a wing in steady motion; this centre is known as the ***centre of independence.***

Determination of critical flutter speed

Consider a wing section of chord ***c*** oscillating harmonically in an airflow of velocity *V*and density ***p*** and having instantaneous displacements, velocities and accelerations of, rotationally, ***a, a′, a″*** and, translationally, y′, y″ , y The oscillation causes a reduction in lift from the steady state lift4 so that, in effect, the lift due to the oscillation acts downwards. The downward lift corresponding to ***y, y′,y″*** is, respectively

$$l_\alpha \rho c V^2 \alpha = L_\alpha \alpha$$

$$l_{\dot{\alpha}} \rho c^2 V \dot{\alpha} = L_{\dot{\alpha}} \dot{\alpha}$$

$$l_{\ddot{\alpha}} \rho c^3 \ddot{\alpha} = L_{\ddot{\alpha}} \ddot{\alpha}$$

In which ***l***a, *la′, la″* are non-dimensional coefficients analogous to the lift-curve slopes insteady motion. Similarly, downward forces due to the translation of the wing sectionoccur and are

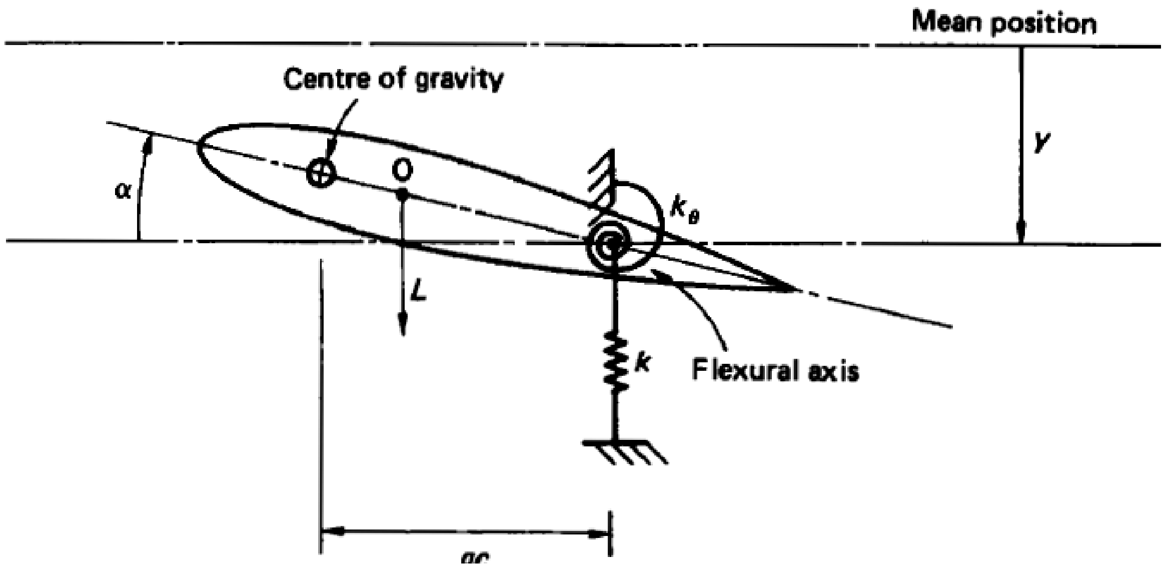
$$l_y \rho c V^2 y/c = L_y y$$

$$l_{\dot{y}} \rho c^2 V \dot{y}/c = L_{\dot{y}} \dot{y}$$

$$l_{\ddot{y}} \rho c^3 \ddot{y}/c = L_{\ddot{y}} \ddot{y}$$

Total aerodynamic lift equals

$$L = L_y y + L_{\dot{y}} \dot{y} + L_{\ddot{y}} \ddot{y} + L_\alpha \alpha + L_{\dot{\alpha}} \dot{\alpha} + L_{\ddot{\alpha}} \ddot{\alpha}$$



Similarly for moment

$$M - I_O \ddot{\alpha} + mgc\ddot{y} - k_\theta \alpha = 0$$

Substituting for these two equations

$$(m - L_{\ddot{y}})\ddot{y} - L_{\dot{y}}\dot{y} + (k - L_y)y - (mgc + L_{\ddot{\alpha}})\ddot{\alpha} - L_{\dot{\alpha}}\dot{\alpha} - L_\alpha \alpha = 0$$
$$(-mgc - M_{\ddot{y}})\ddot{y} - M_{\dot{y}}\dot{y} - M_y y + (I_O - M_{\ddot{\alpha}})\ddot{\alpha} - M_{\dot{\alpha}}\dot{\alpha} + (k_\theta - M_\alpha)\alpha = 0$$

The terms involving ***y*** in the force equation and *a* in the moment equation are knownas ***direct*** terms, while those containing *a* in the force equation and *y* in the momentequation are known as ***coupling*** terms.

Prevention of flutter

We have previously seen that flutter can be prevented by eliminating inertial,aerodynamic and elastic coupling by arranging for the centre of gravity, the centreof independence and the flexural axis of the wing section to coincide.the inertial coupling term **is** mgc + ***My*** in which ***My*** is usually verymuch smaller than mgc. Thus, inertial coupling may be virtually eliminated byadjusting the position of the centre of gravity of the wing section through massbalancing so that it coincides with the flexural **axis,** i.e. gc = 0. The aerodynamiccoupling term ***Myy'***vanishes, as we have seen, when the centre of independencecoincides with the flexural axis. Further, the terms *M,J*and ***Laa'***are very smalland may be neglected so that

$$(m - L_{\ddot{y}})\ddot{y} - L_{\dot{y}}\dot{y} + (k - L_y)y - L_\alpha \alpha = 0$$

$$(I_O - M_{\ddot{\alpha}})\ddot{\alpha} - M_{\dot{\alpha}}\dot{\alpha} + (k_\theta - M_\alpha)\alpha = 0$$

The remaining coupling term *L,a*cannot be eliminated since the vertical forcerequired to maintain flight is produced by wing incidence. The above equation governs the torsional motion of the wing section and contains nocoupling terms so that, since all the coefficients are positive at speeds below the wingsection torsional divergence speed, any torsional oscillation produced, say, by a gustwill decay. In practice it is not always possible to prevent flutter by eliminating coupling terms.

However, increasing structural stiffness, although carrying the penalty of increasedweight, can raise the value of *V,* above the operating speed range. Further, arrangingfor the centre of gravity of the wing section to be as close as possible to and forward ofthe flexural axis is beneficial. Thus, wing mounted jet engines are housed in pods wellahead of the flexural axis of the wing.